

# Bloch Equation

$$\frac{d\vec{M}}{dt} = \underbrace{\vec{M} \times \gamma \vec{B}}_{\text{precession}} - \underbrace{\frac{M_x \hat{i} + M_y \hat{j}}{T_2}}_{T_2 \text{ relaxation}} - \underbrace{\frac{(M_z - m) \hat{k}}{T_1}}_{T_1 \text{ relaxation}}$$

consider  $\vec{B} = B_0 \hat{k}$        $\omega_0 = \gamma B_0$

$$\begin{bmatrix} \dot{M}_x \\ \dot{M}_y \\ \dot{M}_z \end{bmatrix} = \begin{bmatrix} 0 & \omega_0 & 0 \\ -\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$



$$\vec{M}(t) = R_z(\omega_0 t) \vec{M}(0)$$

left handed rotation about z axis  
by angle  $\omega_0 t$

$$\begin{bmatrix} \cos \omega_0 t & \sin \omega_0 t & 0 \\ -\sin \omega_0 t & \cos \omega_0 t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

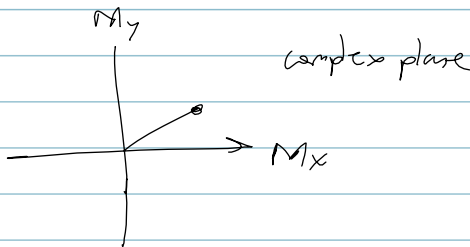
now lets include  $T_2$  &  $T_1$

$$\vec{M}(t) = \begin{bmatrix} e^{-t/T_2} & 0 & 0 \\ 0 & e^{-t/T_2} & 0 \\ 0 & 0 & e^{-t/T_1} \end{bmatrix} R_z(\omega_0 t) \vec{M}(0) + \begin{bmatrix} 0 \\ 0 \\ M_0 (1 - e^{-t/T_1}) \end{bmatrix}$$

referred  $M_z(t) = M_z(0) e^{-t/T_1} + M_0 (1 - e^{-t/T_1})$   
 $T_1$  recover

Lets focus on the received signal, comes from X, y components  
FID

~~M~~  $M \triangleq M_x + i M_y$



Bloch Eqn

$$\frac{dM}{dt} = \frac{dM_x}{dt} + i \frac{dM_y}{dt}$$

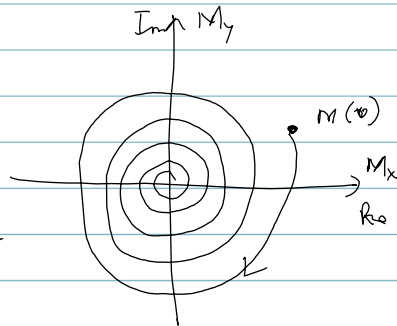
$$= \underbrace{\omega_0 M_y - i \omega_0 M_x}_{\text{precession}} - \underbrace{\frac{1}{T_2} (M_x + i M_y)}_{\text{relaxation}}$$

↗ PVS PAGE  
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$$-i \omega_0 (M_x + i M_y)$$

$$\frac{dM}{dt} = - \left( i \omega_0 + \frac{1}{T_2} \right) M$$

$$M(t) = M(0) e^{-i \omega_0 t} e^{-\frac{1}{T_2} t}$$



lets be more general

$$M(x, y, z, t) = M_x(x, y, z, t) + i M_y(x, y, z, t)$$

$\uparrow$   $\uparrow$   $\uparrow$   
 $x$  component of  
 the magnetization  
 $x$  position

$$\vec{r} = x, y, z$$

$$M(\vec{r}, t)$$

$$B(\vec{r}, t), T_2(\vec{r})$$

$$\frac{dM(\vec{r}, t)}{dt} = - \left[ i(\omega_0 + \Delta\omega(\vec{r}, t)) + \frac{1}{T_2(\vec{r})} \right] M(\vec{r}, t)$$



$$B_z(\vec{r}, t) = B_0 + \Delta B(\vec{r}, t)$$

$$M(\vec{r}, t) = M(\vec{r}, 0) e^{-t/T_2(\vec{r})} e^{-i\omega_0 t} e^{-i \int_0^t \Delta\omega(\vec{r}, t) dt}$$

same as before

time dependent  
phase

$$M_a(\vec{r}, t)$$



$$\Delta w(\vec{r}, t) = \begin{cases} \gamma G_{yx} & \text{linear } G_{yx} \text{ on} \\ \gamma \vec{G} \cdot \vec{r} = \gamma(G_{yx}x + G_{yz}z + G_{zt}) & \text{arbitrary linear gradient} \\ \gamma \vec{G}(t) \cdot \vec{r} & \text{time varying} \end{cases}$$

$$M(\vec{r}, t) = \begin{cases} M_a(\vec{r}, t) e^{-i \gamma G_{yx} x t} \\ M_a(\vec{r}, t) e^{-i \gamma (\vec{G} \cdot \vec{r}) t} \\ M_a(\vec{r}, t) e^{-i \gamma \int_0^t \vec{G}(\tau) \cdot \vec{r} d\tau} \end{cases} \quad * \leftarrow$$

$$\downarrow$$

$$e^{-i 2\pi \left( \frac{\gamma}{2\pi} \int_0^t \vec{G}(\tau) \cdot \vec{r} d\tau \right) \cdot \vec{r}}$$

consider signal received by RF coil

Faraday's law

$$\text{EMF } \mathcal{E} = - \frac{d\Phi}{dt} \quad \mathcal{E} \propto - \frac{dM(\vec{r}, t)}{dt}$$

FCV signal

$$S_r(t) = -K \int_{\text{volume}} - [i\omega_0 + \gamma \vec{G}(t) \cdot \vec{r}] M(\vec{r}, t) dV$$

$\approx i\omega_0$

ignore  $T_z$

use ps expansion \* ←

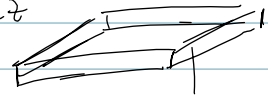
$$s_r(t) = \underbrace{(k_i \omega_0)}_{\text{const}} \int_{\text{slice}} m_0(\vec{r}) e^{-i\omega_0 t} e^{-i 2\pi \left( \frac{\delta}{2\pi} \int_0^t \tilde{u}(r) dr \right) \cdot \vec{r}} dV$$

### Three simplifications

1) continue ignoring  $T_2$

2) assume 2D imaging, excited thin slice  $\Delta z$

$$\text{define } m_0(x, y) = \int_{\Delta z} m_0(x, y, z) dz$$



3) demodulate by  $\omega_0$

$$s(t) \hat{=} s_r(t) e^{+i\omega_0 t} \quad \text{"baseband"}$$

$$s(t) = \iint_{x, y} m_0(x, y) e^{-i 2\pi \left[ \underbrace{\left( \frac{\delta}{2\pi} \int_0^t \tilde{u}_x(r) dr \right)}_{k_x(t)} x + \underbrace{\left( \frac{\delta}{2\pi} \int_0^t \tilde{u}_y(r) dr \right)}_{k_y(t)} y \right]} dx dy$$

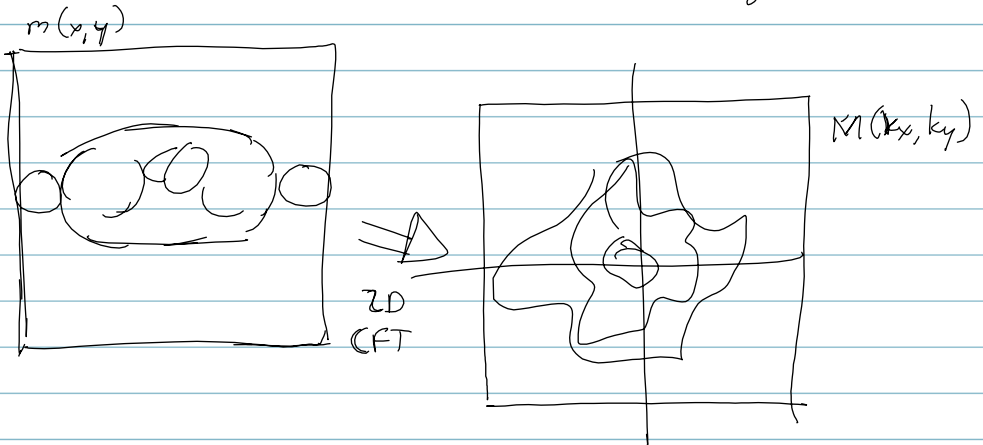
$$= \iint_{x, y} m(x, y) e^{-i 2\pi (k_x(t) x + k_y(t) y)} dx dy$$

$$\Rightarrow \mathcal{F}\{m(x, y)\} \Big|_{\substack{\text{spatial freq} \\ k_x(t), k_y(t)}}$$

$$s(t) = M(k_x(t), k_y(t))$$

$$k_x(t) = \frac{\gamma}{2\pi} \int_0^t u_x(\tau) d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi} \int_0^t u_y(\tau) d\tau$$



sampling  $M(k_x, k_y)$

map of trajectories in the spatial frequency domain  
 $u_x, u_y$  control

